Instance-Optimal Computational Geometry Made Easier and Sensitive to Sortedness

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Introduction

Instance-optimal algorithms achieve performance as good as any correct algorithm on every input instance up to a constant factor with respect to a given measure.

Real-world data often has underlying patterns or structures that are not captured by traditional worst-case analysis. These hidden properties can be leveraged to improve the efficiency of algorithms.

We study how to exploit input *sortedness* in geometric problems.

Motivation & Problem

Previous work focused on *output size* and *spatial distribution* for instance-optimality. However, the role of input sortedness remains unexplored in computational geometry.

- **Sortedness:** Measures how close the input is to being sorted, such as through *inversions*, *removals*, and *runs*.
- **Shannon sequential entropy:** Measures the degree of order.
- **Shannon structural entropy:** Measures the placement and spread.

Our goal: Define a complexity measure that captures sortedness and structure to design and analyze algorithms that adapt to it.

New Complexity Measure

Shannon range-partition entropy combines Shannon sequential and structural entropy and subsumes both.

Given a set, S, of n points, a partition, Π, of the set into disjoint subsets is *respectful* if:

1. **Local property:** Fulfills properties within a subset.
2. **Global compatibility:** Fulfills dependencies between subsets.

The *entropy*, $\mathcal{H}(\Pi)$, of a partition, $\Pi = \{(S_1, R_1), \ldots, (S_t, R_t)\}$, is

$$\mathcal{H}(\Pi) = -\sum_{i=1}^{t} \left(\frac{|S_i|}{n}\right) \log \left(\frac{|S_i|}{n}\right).$$

The *range-partition entropy*, $\mathcal{H}(S)$, of S is the minimum $\mathcal{H}(\Pi)$ over all respectful partitions.

References

[1] Peyman Afshani, Jérémy Barbay, and Timothy M. Chan. Instance-optimal geometric algorithms. *Journal of the ACM*, 64(1):A3:1–A3:38, March 2017.

[2] Nicolas Auger, Vincent Jugé, Cyril Nicaud, and Carine Pivoteau. On the worst-case complexity of TimSort, 2019. Previously announced at ESA 2018.

[3] David G. Kirkpatrick and Raimund Seidel. Output-size sensitive algorithms for finding maximal vectors. In *1st ACM Symposium on Computational Geometry (SoCG)*, page 89–96, 1985.

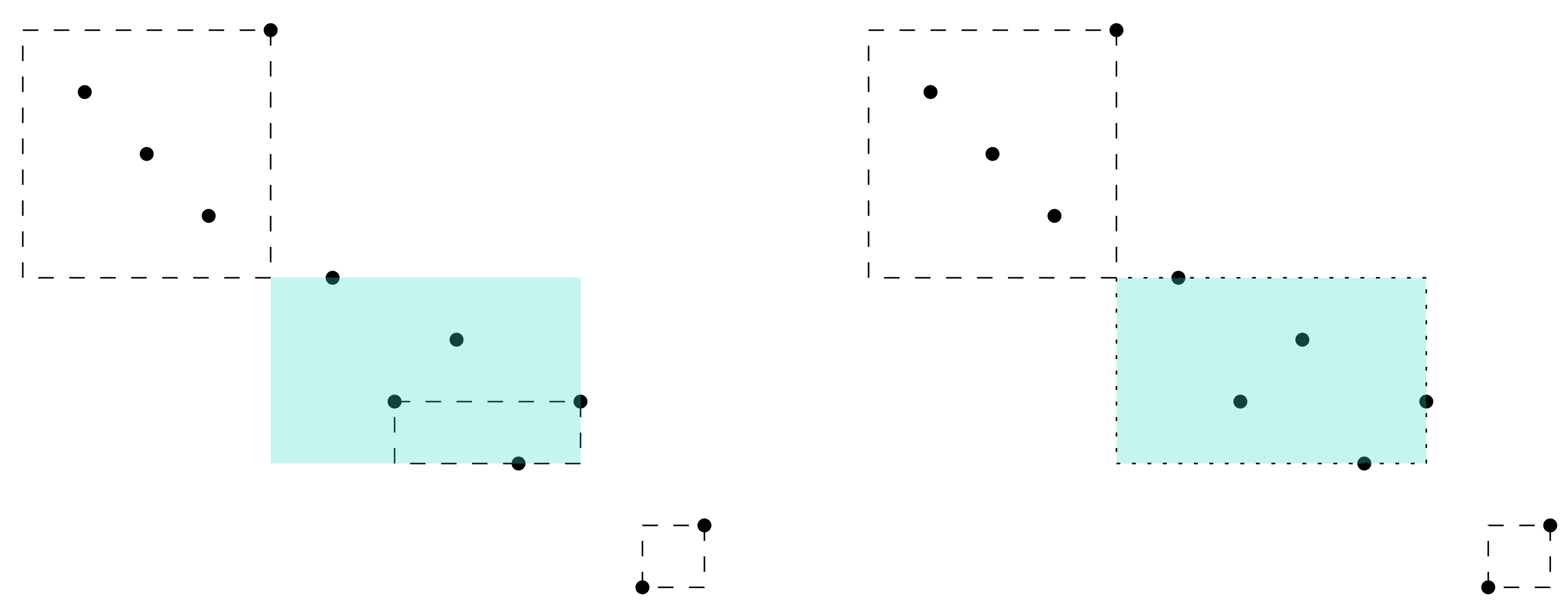
[4] David G. Kirkpatrick and Raimund Seidel. The ultimate planar convex hull algorithm? *SIAM Journal on Computing*, 15(1):287–299, 1986.

2D Maxima Set

Problem: Find the subset of points that are not dominated by any other point (no other point has both greater x- and y-coordinates).

Algorithm: Divide-and-conquer algorithm [1, 3]. Before recursively solving a subset, check if the points are sorted; if so, compute the maxima set in linear time.

Analysis: Leverages structure and sortedness. Runs in $O(n(1 + \mathcal{H}(S)))$ time, where $\mathcal{H}(S)$ is the range-partition entropy of S.



(a) (b)

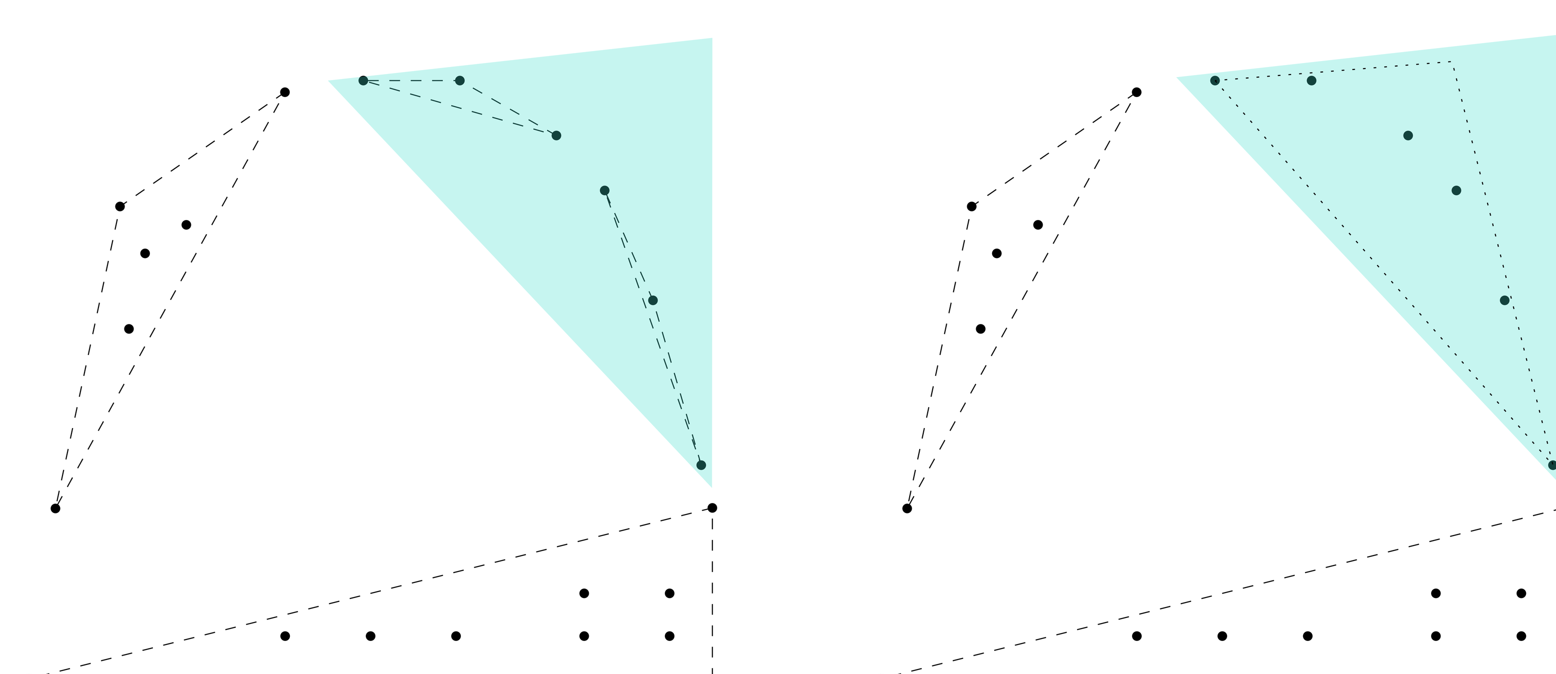
Figure 1. Respectful partitions for maxima set. The points in the blue shaded rectangle are sorted among themselves. (a) A respectful partition without using sorted subsets. (b) A respectful partition using both types of sets, leveraging sortedness.

2D Convex Hull

Problem: Find the smallest convex polygon enclosing all points.

Algorithm: Divide-and-conquer algorithm [1, 4]. Before recursively solving a subset, check if the points are sorted; if so, compute the convex hull in linear time.

Analysis: Leverages structure and sortedness. Runs in $O(n(1 + \mathcal{H}(S)))$ time, where $\mathcal{H}(S)$ is the range-partition entropy of S.



(a) (b)

Figure 2. Respectful partitions for convex hull. The points in the blue shaded triangle are sorted among themselves. (a) A respectful partition without using sorted subsets. (b) A respectful partition using both types of sets, leveraging sortedness.

3D Convex Hull

Problem: Find the smallest convex polyhedron enclosing all points.

Algorithm: Iteratively partition and prune [1]. Partitioning via eight-sectioning in expected linear time with random sampling and combinatorial partitioning. Recursive partitioning runs in $O(n \log n)$ time.

Analysis: Leverages structure with improved partitioning. Runs in $O(n(1+\mathcal{H}(S)))$ time, where $\mathcal{H}(S)$ is the range-partition entropy of S.

Lower Envelope

Problem: Find the vertical point-wise minimum of a set of disjoint monotone line segments.

Algorithm: Stack-based mergesort algorithm with redefined weights.

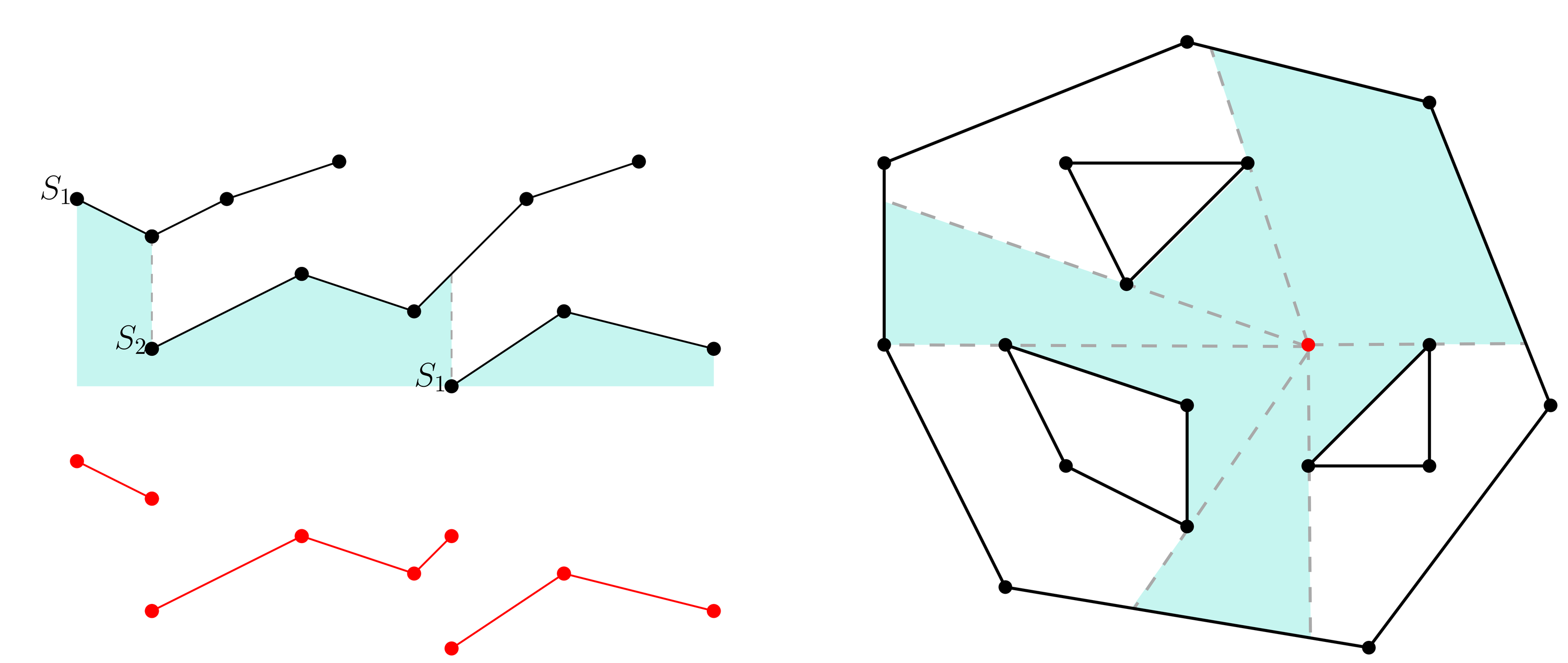
Analysis: Leverages monotonic runs similar to TimSort [2]. Runs in $O(n(1+\mathcal{H}(S)))$ time, where $\mathcal{H}(S)$ is the range-partition entropy of S.

Visibility Polygon

Problem: Find the radial point-wise minimum, the visible region, from a point inside a convex polygon.

Algorithm: Stack-based mergesort algorithm with redefined weights.

Analysis: Reduction from lower envelope. Runs in $O(n(1 + \mathcal{H}(S)))$ time, where $\mathcal{H}(S)$ is the range-partition entropy of S.



(a) (b)

Figure 3. (a) Merging two sets of disjoint monotone chains. S_1 has two sequences, S_2 has one. (b) The visibility polygon of a point among disjoint convex chains.

Conclusion

Sortedness is a powerful but overlooked property in instance-optimal analysis. By leveraging sortedness, we can design algorithms that are more efficient than traditional worst-case approaches. We demonstrated how algorithms for classical geometric problems can benefit from recognizing and exploiting input sortedness.

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